# Orbital Mechanics Elliptic Orbits

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**For Mathematica® Code Visit:**
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# Contents

## List of Figures

# Chapter 1

## Introduction

Elliptic orbits are covered with derivations and also Mathematica® code to illustrate the velocity vector as, for example, a satellite orbits the Earth. Also covered, with Mathematica® code included, is the position of a satellite with specified time (time of flight).

# Chapter 2

## The Ellipse and Key Equations

### 2.1 Geometry

Figure 1 shows the geonmetry of an ellipse. It is important to grasp the relationships between the various parameters in an ellipse.

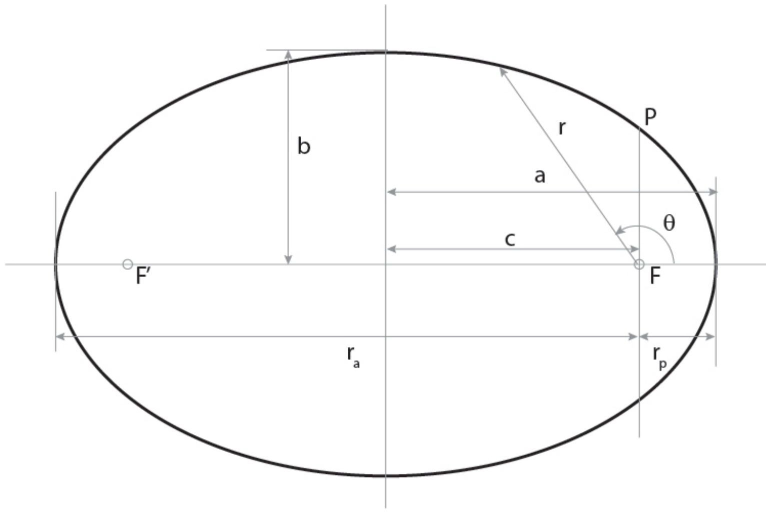


Figure 1: Ellipse

To start, note that the equation for an ellipse in polar coordinates is:

$$r = \frac{p}{1 + e\cos\theta} \tag{1}$$

where $e$ is the eccentricity and is defined as:

$$e = \frac{c}{a} \tag{2}$$

Now from Figure 1:

$$a = \frac{r_p + r_a}{2} \tag{3}$$

Also

$$c = \frac{r_a - r_p}{2} \tag{4}$$

From Figure 1, setting $\theta = 0$,

$$r_p = \frac{p}{1 + e} \tag{5}$$

and setting $\theta = \pi$,

$$r_a = \frac{p}{1-e} \tag{6}$$

So,

$$a = \frac{r_p + r_a}{2} = \frac{1}{2}p(\frac{1}{1+e} + \frac{1}{1-e}) = \frac{p}{1-e^2} \tag{7}$$

Or,

$$p = \frac{a}{1-e^2} \tag{8}$$

Now lets relate $e$ to $a$ and $b$. Recall,

$$e = \frac{c}{a} \tag{9}$$

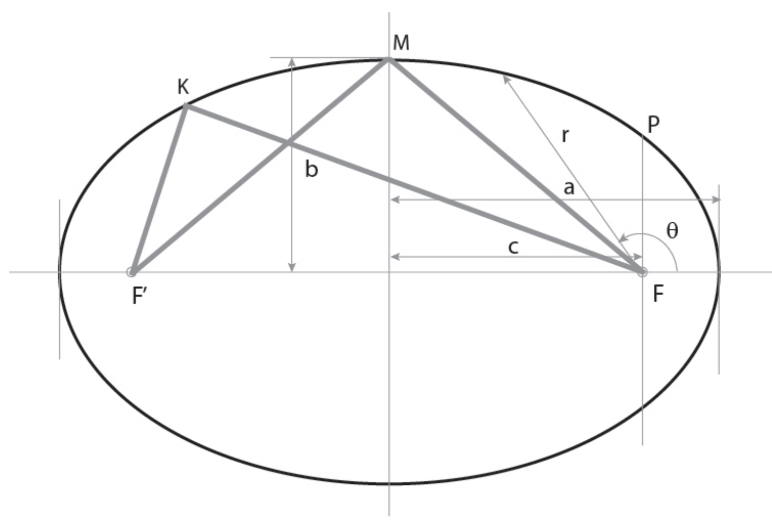


Figure 2: Ellipse Definition

From Figure 2, which is the definition of an ellipse, the two sums hold:

$$F'K + FK = 2a \tag{10}$$

and

$$F'M + FM = 2a \tag{11}$$

Subsequently ,

$$FM = F'M = a \tag{12}$$

Therefore,

$$a^2 = b^2 + c^2 \tag{13}$$

So that,

$$e = \frac{\sqrt{a^2 - b^2}}{a} \tag{14}$$

Thus the eccentriciy has been related to the semimajor axis $a$ and the semiminor axis $b$.

## 2.2 Summary of Equations

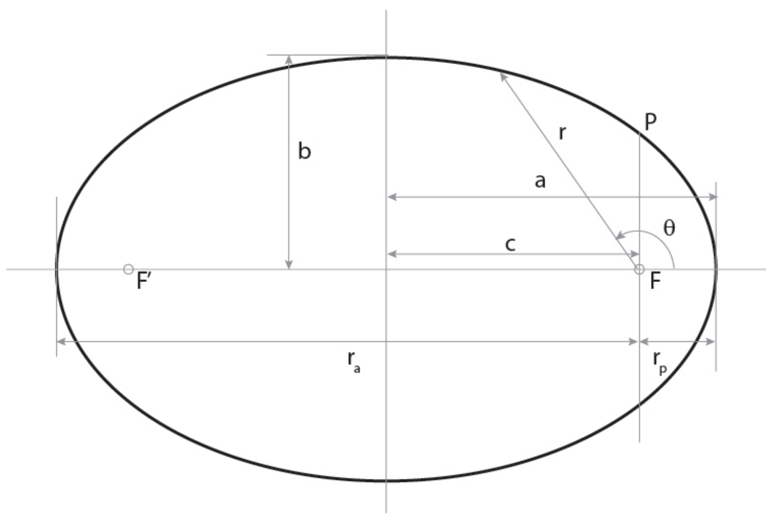


Figure 3: Ellipse Reference

$$e = \frac{c}{a} \tag{15a}$$

$$e = \frac{\sqrt{a^2 - b^2}}{a} \tag{15b}$$

$$a = \frac{r_p + r_a}{2} \tag{15c}$$

$$c = \frac{r_p - r_a}{2} \tag{15d}$$

$$b = \sqrt{a^2 - c^2} \tag{15e}$$

$$b = a\sqrt{1 - e^2} \tag{15f}$$

$$p = a(1 - e^2) \tag{15g}$$

$$r_p = \frac{p}{1 + e} \tag{15h}$$

$$r_a = \frac{p}{1 - e} \tag{15i}$$

$$r = \frac{p}{1 + e\cos\theta} \tag{15j}$$

# Chapter 3

## Derivation of Equations for Elliptic Orbits

The equations governing the motion of an object of mass $m$ and an object of mass $M$ will be derived. The results of this chapter are summerized in the following chapters for reference. The derivation of the equations of motion are not trivial. We have included every single step with explanations and illustrations.

Many authors have presented the derivation of the equations for orbital mechanics. Many referencing [1] which we shall also benefit from. See the work of [2],[6], and [4] as well as articles on Wikipedia. Also [4] is freely available on the Internet. He references [1]. The Course Notes in [7] is a great reference ( also freely available). There is a reason we have written this chapter, so a comprehensive, connected derivation is provided with no ambiguities.

Of course, Kepler, Newton, and Leibniz are responsible for the equations and laws of orbital mechanics but much credit is also due to Tycho Brahe's astronomical observations as well as the Astronomers throughout history going back to the Babolonians. See [1] for great historical notes on Kepler.

In this paper, we derive the equations of motion for the two body system and provide all the details.

In Figure 4 for the two body system the only force on mass $m$ is the gravitational force between mass $m$ and mass $M$ which is:

$$F_m = mMG\frac{\mathbf{r}}{r^3} \tag{16}$$

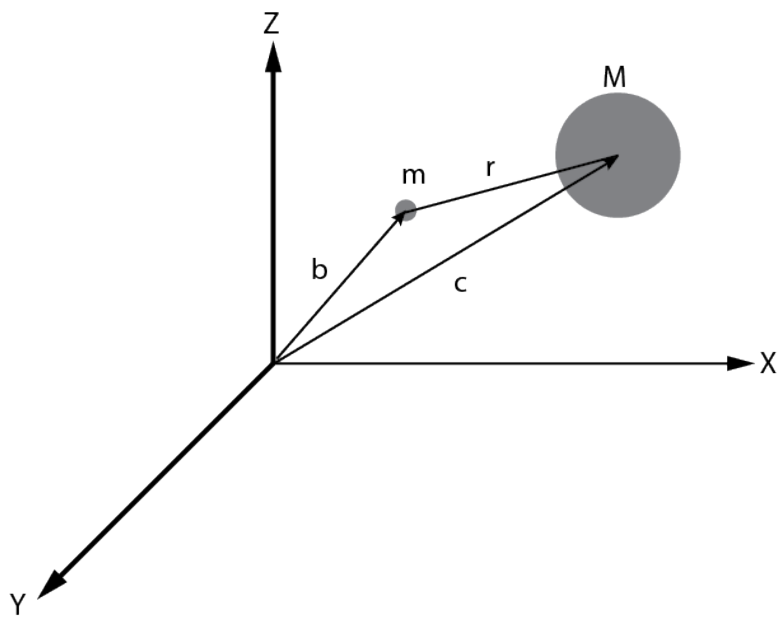


Figure 4: Two Body System

Note that the unit vector along $\mathbf{r}$ is $\frac{\mathbf{r}}{r}$. The gravitational force is inversely related to $r^2$.

The force on $M$ is:

$$F_M = -mMG\frac{\mathbf{r}}{r^3} \tag{17}$$

Now, in the inertial frame.

$$m\ddot{\mathbf{b}} = mMG\frac{\mathbf{r}}{r^3} \tag{18}$$

and

$$m\ddot{\mathbf{c}} = -mMG\frac{\mathbf{r}}{r^3} \tag{19}$$

The vector $\mathbf{r}$ is:

$$\mathbf{r} = \mathbf{b} - \mathbf{c} \tag{20}$$

So,

$$\ddot{\mathbf{r}} = \ddot{\mathbf{b}} - \ddot{\mathbf{c}} \tag{21}$$

Therefore, subtracting (19) from (18) we obtain:

$$\ddot{\mathbf{r}} = -G(M - m)\frac{\mathbf{r}}{r^3} \tag{22}$$

If we set the origin of the Inertial Frame at the center of the assumed spherially symmetric and massive object with mass $M$ and assuming $m \ll M$, then,

$$\ddot{\mathbf{r}} + GM\frac{\mathbf{r}}{r^3} = 0 \tag{23}$$

## 3.1 Conservation of Energy

The following derivation follows [2] and [6]. The conservation of energy can also be found in [4] which references [1].

Let $\mu = GM$. Then (23) becomes:

$$\ddot{\mathbf{r}} + \mu\frac{\mathbf{r}}{r^3} = 0 \tag{24}$$

Form the dot product of (24) with $\dot{\mathbf{r}}$,

$$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \mu\frac{\dot{\mathbf{r}} \cdot \mathbf{r}}{r^3} = 0 \tag{25}$$

Now,

$$\frac{d(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})}{dt} = 2\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} \tag{26}$$

and

$$\frac{d(\mathbf{r} \cdot \mathbf{r})}{dt} = 2\mathbf{r} \cdot \dot{\mathbf{r}} \tag{27}$$

So (25) can be written:

$$\frac{d(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})}{dt} + \frac{\mu}{r^3} \frac{d(\mathbf{r} \cdot \mathbf{r})}{dt} = 0 \tag{28}$$

Integrating (29) with respect to time $t$ we obtain:

$$\dot{r}^2 + \frac{\mu}{r} = \epsilon \tag{29}$$

From [2], here $\dot{r}^2$ is the specific kinetic energy and $\frac{\mu}{r}$ is the specific potential energy of the object with mass $m$. Also see [4].

So as the object with mass $m$ gains in kinetic energy (speed) its potential energy decreases (distance to mass $M$ ). Also as its potential enegy inceases ( distance to mass $M$ increases) its speed decreases. This will be examined in detail in a later chapter. Note that the object with mass $M$ is fixed in location ( the assumption that $m \ll M$).

## 3.2 Conservation of Angular Momentum

We know that angular momentum is ralated to $\mathbf{r} \times \dot{\mathbf{r}}$ so it makes sense to cross multiply (24) by $\mathbf{r}$.

$$\mathbf{r} \times \ddot{\mathbf{r}} + \mu \mathbf{r} \times \frac{\mathbf{r}}{r^3} = 0 \tag{30}$$

Thus,

$$\mathbf{r} \times \ddot{\mathbf{r}} = 0 \tag{31}$$

Since $\mathbf{r} \times \mathbf{r} = 0$.
However,

$$\frac{d(\mathbf{r} \times \dot{\mathbf{r}})}{dt} = \mathbf{r} \times \ddot{\mathbf{r}} + \dot{\mathbf{r}} \times \dot{\mathbf{r}} \tag{32}$$

Note that $\dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$.
Which leads to,

$$\mathbf{r} \times \ddot{\mathbf{r}} = \frac{d(\mathbf{r} \times \dot{\mathbf{r}})}{dt} \tag{33}$$

So,

$$\frac{d(\mathbf{r} \times \dot{\mathbf{r}})}{dt} = 0 \tag{34}$$

Or,

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{H} \tag{35}$$

where $\mathbf{H}$ is a constant vector. This means that $\mathbf{r}$ and $\dot{\mathbf{r}}$ are in the same plane throughout the motion of the object with mass $m$.

Now, following [2], cross multiplying (24) by $\mathbf{H}$,

$$\mathbf{H} \times \ddot{\mathbf{r}} + \mathbf{H} \times \mu \frac{\mathbf{r}}{r^3} = 0 \tag{36}$$

Or,

$$\ddot{\mathbf{r}} \times \mathbf{H} = \mu \frac{\mathbf{H} \times \mathbf{r}}{r^3} \tag{37}$$

Now,

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{H})}{dt} = \ddot{\mathbf{r}} \times \mathbf{H} + \mathbf{r} \times \dot{\mathbf{H}} \tag{38}$$

But $\mathbf{H}$ is a constant vector so $\dot{\mathbf{H}}$=0.

Finally,

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{H})}{dt} = \mu \frac{\mathbf{H} \times \mathbf{r}}{r^3} \tag{39}$$

Now, based on (35) :

$$\mathbf{H} \times \mathbf{r} = (\mathbf{r} \times \mathbf{v}) \times \mathbf{r} \tag{40}$$

Where $\mathbf{H} = \mathbf{v} \times \mathbf{r}$ is the angular momemntum.

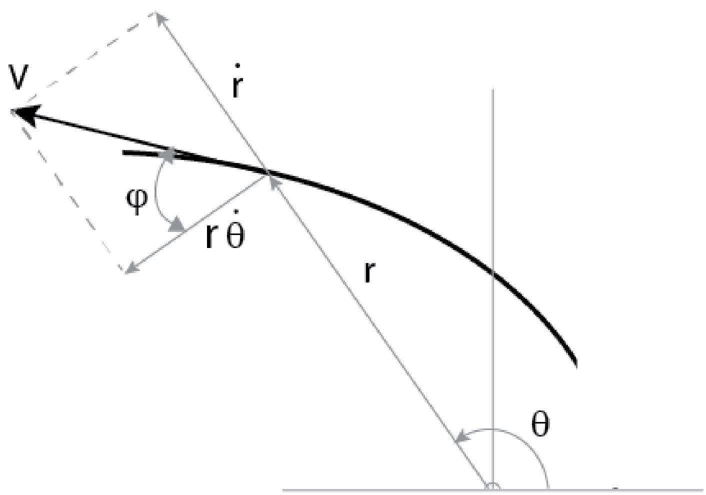


Figure 5: Velocity Vector

We will use the vector relationship:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{41}$$

So that

$$-(\mathbf{r} \times \mathbf{v}) \times \mathbf{r} = \mathbf{r} \times (\mathbf{r} \times \mathbf{v}) = (\mathbf{r} \cdot \mathbf{v})\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\mathbf{v} \tag{42}$$

Lets focus on the equation:

$$(\mathbf{r} \cdot \mathbf{v})\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\mathbf{v} \tag{43}$$

With reference to Figure 5, we will examine the term $\mathbf{r} \cdot \mathbf{v}$.

$$\mathbf{r} \cdot \mathbf{v} = rv\cos(\phi) \tag{44}$$

Now from Figure 5,

$$\dot{r} = v\cos(\phi) \tag{45}$$

Elliminate $\cos(\phi)$ from (44) and (45) we get,

$$\mathbf{r} \cdot \mathbf{v} = r\dot{r} \tag{46}$$

The above procedure for obtaining (46) was based on [6].
Thus we can write,

$$-(\mathbf{r} \times \mathbf{v}) \times \mathbf{r} = (\mathbf{r} \cdot \mathbf{v})\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\mathbf{v} = r\dot{r}\mathbf{r} - r^2\mathbf{v} \tag{47}$$

For reference we show equation (39)

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{H})}{dt} = \mu \frac{\mathbf{H} \times \mathbf{r}}{r^3} \tag{48}$$

Which we can write, based on (47), as:

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{H})}{dt} = -\mu \frac{r\dot{r}\mathbf{r} - r^2\mathbf{v}}{r^3} \tag{49}$$

Or,

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{H})}{dt} = -\mu(\frac{\dot{r}\mathbf{r}}{r^2} - \frac{\mathbf{v}}{r}) \tag{50}$$

Note that $\frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2}$. Therefore we recognize that,

$$\frac{d}{dt}(\frac{\mathbf{r}}{r}) = -\frac{\dot{r}\mathbf{r}}{r^2} + \frac{\dot{\mathbf{v}}}{r} \tag{51}$$

Note that $\mathbf{v} = \dot{\mathbf{r}}$, see [5] , and we must distinguish between $\dot{\mathbf{r}}$ and $\dot{r}$.
Finally,

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{H})}{dt} = \frac{d}{dt}(\frac{\mathbf{r}}{r}) \tag{52}$$

Integrating both sides of (52) we obtain:

$$\dot{\mathbf{r}} \times \mathbf{H} = \frac{\mathbf{r}}{r} + \mathbf{B} \tag{53}$$

The vector $\mathbf{B}$ is a constant of integration.
Following [6] and [2], dot multiply both sides of (53) by $\mathbf{r}$

$$\mathbf{r} \cdot \dot{\mathbf{r}} \times \mathbf{H} = \mathbf{r} \cdot \frac{\mathbf{r}}{r} + \mathbf{r} \cdot \mathbf{B} \tag{54}$$

Following [6], we use the following identity:

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c} \tag{55}$$

So that,

$$\mathbf{r} \cdot \dot{\mathbf{r}} \times \mathbf{H} = \mathbf{r} \times \dot{\mathbf{r}} \cdot \mathbf{H} = \mathbf{H} \cdot \mathbf{H} = H^2 \tag{56}$$

Now,

$$\mathbf{r} \cdot \frac{\mathbf{r}}{r} + \mathbf{r} \cdot \mathbf{B} = r + rBcos(\psi) \tag{57}$$

Which leads to, based on (54),

$$H^2 = \mu r + \mu Bcos(\psi) \tag{58}$$

Finally,

$$r = \frac{\frac{H^2}{\mu}}{1 + \frac{B}{\mu}\cos(\psi)} \tag{59}$$

See Chaper 1 for the mathematical treatment of this equation for the case that $e = \frac{B}{\mu} < 1$ which is an Ellipses. See Figure 2.

In general (59) describes conic sections. For conic sections with various mathematical representations and animations, see [3].

A comment on the angle between **B** and **r**. There is every reason to assume that this angle can be taken as $\theta$ as in Figure 5 especially since the vector **B** is a constant of integration.

### 3.3 Orbit Equation

Define,

$$p = \frac{H^2}{\mu} \tag{60}$$

$p$ is called the "semi parameter" Also, define,

$$e = \frac{B}{\mu} \tag{61}$$

$e$ is called the eccentricty.

So (59) can be written as

$$r = \frac{p}{1 + e\cos(\theta)} \tag{62}$$

In the following chapters $\theta$ is called the True Anomaly. In a later chapter we will show a graph using Mathematica® code where the velocity vector is shown as a function of the True Anomaly $\theta$.

### 3.4 Velocity

Since the angular momentum is constant:

$$H = r_a v_a \tag{63}$$

$$H = r_p v_p \tag{64}$$

So the Energy is from (29) substituting for $v_p$ ( we are using $E$ for energy)

$$E = \frac{v^2}{2} - \frac{\mu}{r} = \frac{H^2}{2r_p^2} - \frac{\mu}{r_p} \tag{65}$$

At $\theta = 0$, we have $r_p = \frac{p}{1+e}$. But $p = a(1-e^2)$ so

$$r_p = \frac{a(1-e^2)}{1+e} = a(1-e) \tag{66}$$

From $r_p = a(1-e)$ and from (60) $H = \sqrt{\mu p} = \sqrt{\mu a(1-e^2)}$ so,

$$E = \frac{\mu a(1-e^2)}{2a^2(1-e)^2} - \frac{\mu}{a(1-e)} \tag{67}$$

For $e \neq 0$, i.e. not a parabolic path, this reduces to,

$$E = -\frac{\mu}{2a} \tag{68}$$

Since $E$ is constant, then from (65) substituing for $E$ and rearranging,

$$v^2 = \mu(\frac{2}{r} - \frac{1}{a}) \tag{69}$$

### 3.5 Elliptic Orbit Period

Recall that the transverse component of the velocity vector was $v\cos(\phi)$ from Figure 5. So it is this component that will lead us to compute the Period as it relates to the rate at which the object is moving. Following [6] , we can express the transverse component of the velocity vector from Figure 5 as $r\dot{\theta}$. Now the constant angular momentum is $H = r\dot{\theta}r$. Note that the the $\dot{r}$ component does not contribute to $H$ in $H = \mathbf{r} \times \mathbf{v}$. Or,

$$H = r^2 \frac{d\theta}{dt} \tag{70}$$

Which can be rearranged as,

$$dt = \frac{r^2 d\theta}{H} \tag{71}$$

Now this is interesting. The area swept by $d\theta$ is $dA = \frac{1}{2}r(rd\theta) = \frac{1}{2}r^2 d\theta$. So if we substitute for $r^2\theta$ in (71 ) we obtain:

$$dt = 2\frac{dA}{H} \tag{72}$$

Integrating over $\theta$ from 0 to $2\pi$ gives,

$$P = T = 2\frac{A_{ellipse}}{H} \tag{73}$$

The area of the Ellipse (See Figure 1), is

$$A_{ellipse} = \pi ab \tag{74}$$

Substituting into (75),

$$P = T = 2\frac{\pi ab}{H} \tag{75}$$

Now $p = \frac{H^2}{\mu}$. See equation (60). So $H = \sqrt{p\mu}$. Also $p = a(1 - e^2)$ so,

$$P = T = 2\frac{\pi ab}{\sqrt{p\mu}} = 2\frac{\pi ab}{\sqrt{a(1 - e^2)\mu}} \tag{76}$$

Now $b = a\sqrt{1 - e^2}$. So,

$$P = 2\frac{\pi a^2 \sqrt{1 - e^2}}{\sqrt{a(1 - e^2)\mu}} \tag{77}$$

$$P = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{78}$$

## 3.6 Summary of Equations for Position and Velocity

Constants

$$\mu = GM \tag{79a}$$
$$G = 6.67430 \times 10^{-11} N - m^2 \cdot kg^{-2} \tag{79b}$$
$$M_{Earth} = 5.97221024 kg \tag{79c}$$
$$\mu_{Earth} = 3.986004415 \times 10^5 \frac{km^3}{s^2} \tag{79d}$$

Equations

$$p = \frac{H^2}{\mu} \tag{80a}$$
$$p = a(1 - e^2) \tag{80b}$$
$$r = \frac{p}{1 + e\cos(\theta)} \tag{80c}$$
$$v = \sqrt{\mu(\frac{2}{r} - \frac{1}{a})} \tag{80d}$$
$$P = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{80e}$$

# Chapter 4

## Earth and Jupiter Orbits

## 4.1 An Earth Orbit

The parameters used for the Satellite are from [6] and are summarized below:

$$
\boxed{
\begin{aligned}
\mu &= 3.986004415 * 10^5 && (81a)\\
e &= 0.6 && (81b)\\
p &= 20,410\ km && (81c)\\
a &= 31890\ km && (81d)\\
b &= 25512\ km && (81e)\\
r_p &= 12756\ km && (81f)\\
r_a &= 51024\ km && (81g)\\
c &= 19134\ km && (81h)\\
EarthRadius &= 6371\ km && (81i)
\end{aligned}
}
$$

Constants

$$
\boxed{
\begin{aligned}
\mu &= GM && (82a)\\
G &= 6.67430 \times 10^{-11} N - m^2 \cdot kg^{-2} && (82b)\\
M_{Earth} &= 5.9722 \times 10^{24} kg && (82c)\\
\mu_{Earth} &= 3.986004415 \times 10^5 \frac{km^3}{s^2} && (82d)
\end{aligned}
}
$$

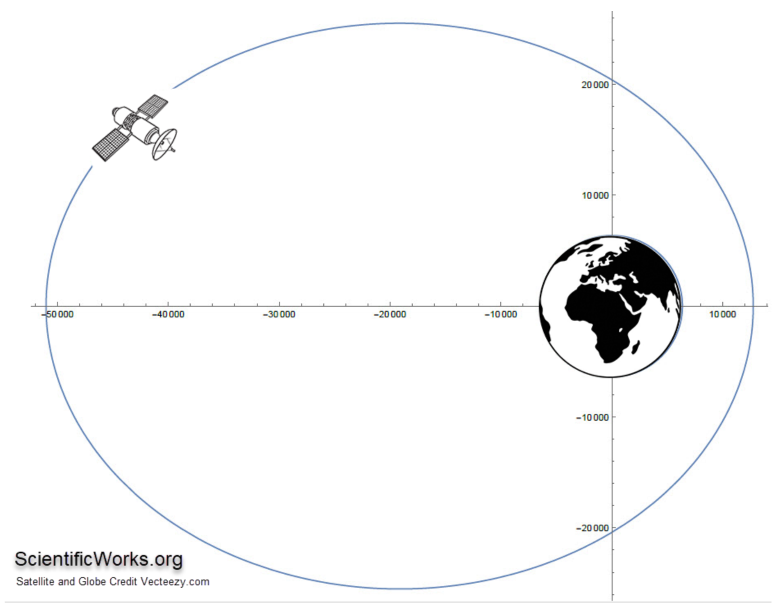


Figure 6: Elliptic Earth Orbit

Table 1: Space Station Calculated Orbit Parameters

| Item | Parameter | Value | Units |
|---|---|---|---|
| 1 | $mass_{Earth}$ Constant | $5.97 \times 10^{24}$ | kg |
| 2 | $\mu$ Constant | $3.98603 \times 10^{5}$ | $km^3 s^{-2}$ |
| 3 | Period( P) Specified | 0.064513889 | Days |
| 4 | Eccentricity(e) Specified | 0.0008051 | |
| 5 | a | 6796.9754 | km |
| 6 | p | 6796.9710 | km |
| 7 | b | 6796.9732 | km |
| 8 | c | 5470.427305 | km |
| 9 | $r_p$ | 6791.5032 | km |
| 10 | $r_a$ | 6800.188 | km |
| 11 | Velocity at r=$r_p$ | 7665.38 | m/s |
| 12 | Velocity at $r = r_p$ | 27604.555 | km/H |

## 4.2 Space Station

The specified parameter for the Space Station is the Period which is $P = 92.9$ minutes and an eccentricity $e = 0.0008051$. Below we calculate the orbital parameters.

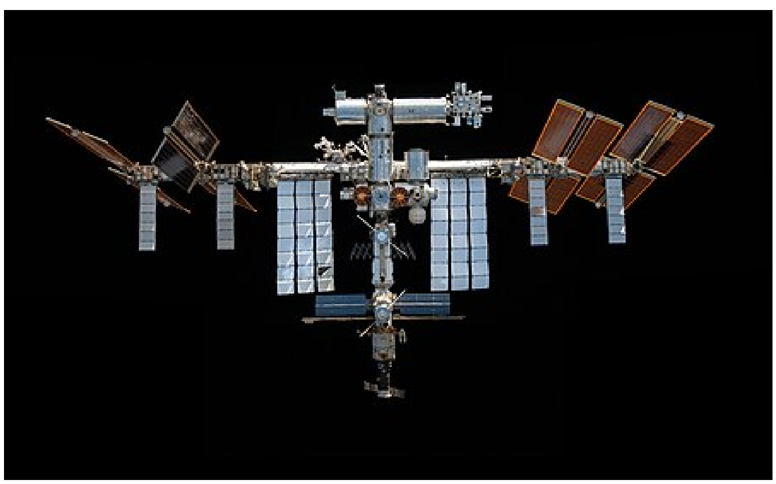

Figure 7: Space Station Wikipedia

Note that the height of the space station at $r_p$ is:

$$h_{spacestation} = r_p - Radius_{Earth} = 6791.5 - 6378 = 420.5\ km \tag{83}$$

Figure 8 shows the computed and graphed orbit with Earth using Mathematica® code.

See the section on Juno's Jupiters Orbit for the Mathematica® code.

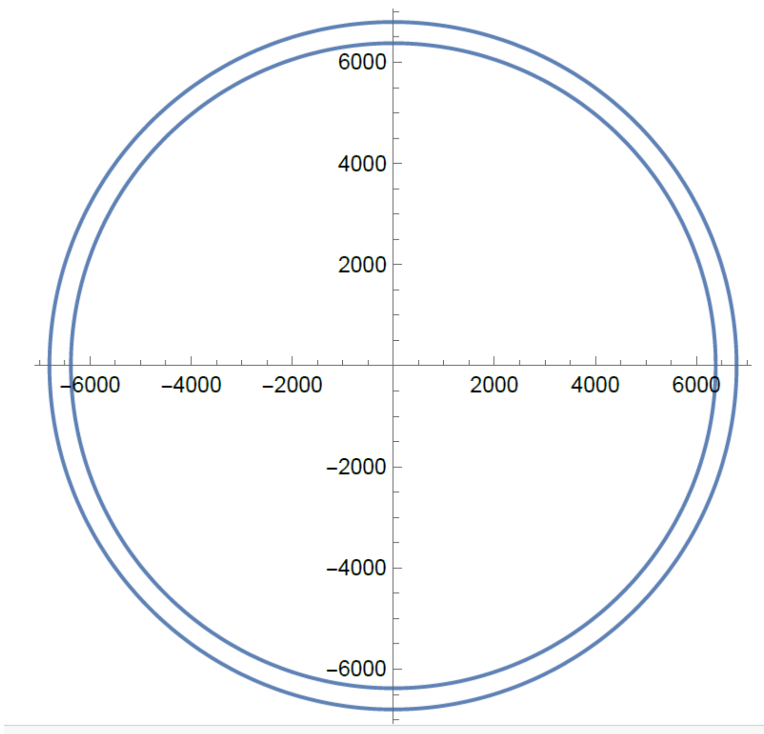


Figure 8: Spacestation Mathematica® Orbit Calculation, Units in Km

## 4.3 Juno Spacecraft Jupiter Orbit

Juno is a NASA spacecraft operated by JPL. It is orbiting Jupiter. In this section we will calculate Juno's Jupiter Orbit. The orbit is also plotted with Mathematica® code, as well as Jupiter itself. The velocity vector of the orbit is also plotted with Mathematica® code. Figure 9 shows the spacecraft.

Figure 9: Juno Spacecraft Wikipedia

Constants

$$\mu = GM \tag{84a}$$
$$G = 6.67430 \times 10^{-11} N - m^2 \cdot kg^{-2} \tag{84b}$$
$$M_{Jupiter} = 1.898 \times 10^{27} kg \tag{84c}$$
$$\mu_{Jupiter} = 1.26678 \times 10^{17} \frac{km^3}{s^2} \tag{84d}$$
$$Radius = 71,600 km \tag{84e}$$

The period for Juno's orbit is 11.07 days. So we can calculate $a$ based on (85).

$$a^3 = \frac{P^2 \mu}{(2\pi)^2} \tag{85}$$

$$a = 1,431,819.63\ km \tag{86}$$

Table 2: Juno Jupiter Orbit Calculated Parameters

| Item | Parameter | Value | Units |
|---|---|---|---|
| 1 | $mass_{Jupiter}$ Constant | $1.9 \times 10^{27}$ | kg |
| 2 | $\mu$ Constant | $1.26678 \times 10^{17}$ | $km^3 s^{-2}$ |
| 3 | Period( P) Specified | 11.07 | Days |
| 4 | Eccentricity(e) Specified | 0.68 | |
| 5 | a | 1,431,819.6 | km |
| 6 | p | 769,746.2 | km |
| 7 | b | 1,049,827.5 | km |
| 8 | c | 973,637.35 | km |
| 9 | $r_p$ | 458,183 | km |
| 10 | $r_a$ | 2,405,457 | km |
| 11 | Velocity at r=$r_p$ | 21552 | m/s |
| 12 | Velocity at $r = r_p$ | 77,587,000 | km/H |

The Orbits eccentricity is:

$$e = 0.68 \tag{87}$$

So $p = a(1 - e^2)$ is:

$$p = 769746km \tag{88}$$

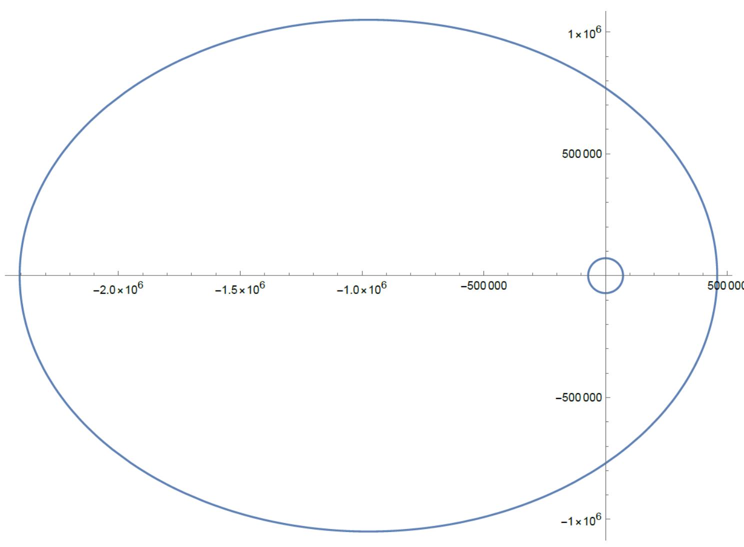


Figure 10: Juno Jupiter Orbit

Mathematica® Code:

```
mu   = 3.986004415*10^5
e  = 0.68
p  = 769843.2
a  = 1432000
b  = 1049959.743
rp  = 2405760
ra  = 2405760
c  = 973760
jupiterRadius = 71600
period = 2*Pi*Sqrt[a*a*a/mu]/3600
Show[PolarPlot[p/(1 + e*Cos[nu]), \{nu, 1, 2.5*Pi\}],
PolarPlot[jupiterRadius, \{nu, 1, 2.5*Pi\}]]
Plot[Sqrt[mu*(2/(p/(1 + e*Cos[nu]) - 1/a))],
 \{nu, 1, 2.5*Pi\}]
```

We also plot the velocity vector. Links to the code is provided in a later Chapter.

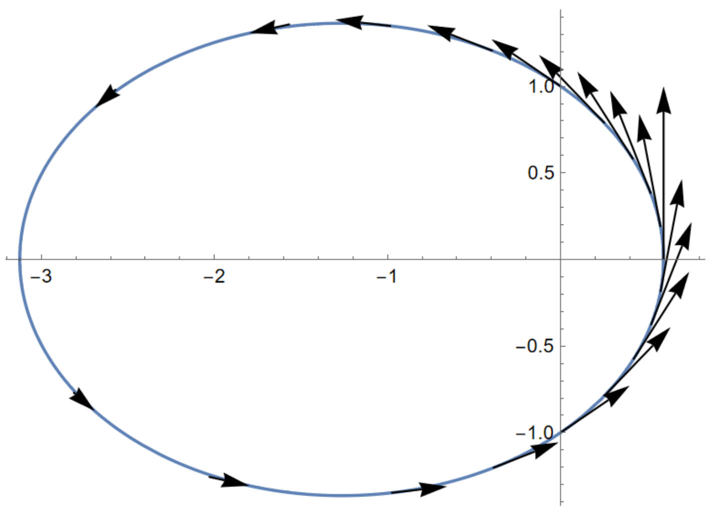


Figure 11: Juno Jupiter Orbit Velocity Vector

# Chapter 5

## Orbit Position as a Function of Time for Elliptic Orbits

This chapter covers aspects of predicting the position of an object in orbit by specifying the time. The position of, for example a satellite, as it orbits, given the angle $\theta$ is straight forward. However, predicting the position of the satellite by specifiying time is a whole different matter.

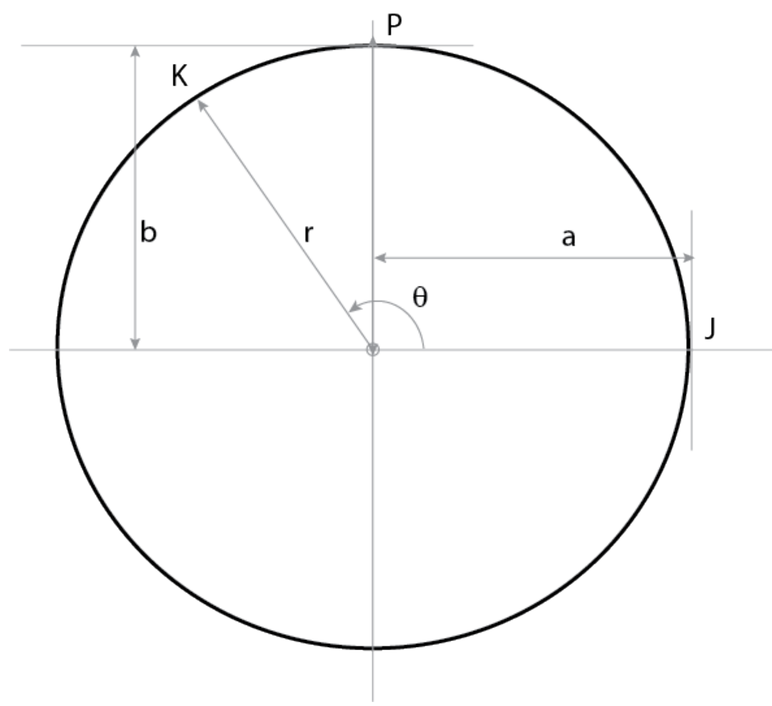


Figure 12: Circular Orbit

Consider a circular orbit as shown in Figure 12. In this case, if we specify the position with the True Analmoly $\theta$ the position at $K$ is known. Now, if we start out at the point where $\theta = 0$, $J$ in Figure 12, and call that time $t = 0$, then, for example a satellite at $K$ will return to position $J$ at $\theta = 0$ after $T$ seconds where $T$ is the orbit period.

The orbit period $T$ is calculated in (89), and is true for both Elliptic and Circular Orbits. In the case of a Circular Orbit $r = a$ and $e = 0$.

$$P = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{89}$$

So to predict the position of the Satellite at $K$ corresponding to a specified time $t$ we need to determined $\theta$ . But for a Circular Orbit,

$$\theta = t\frac{2\pi}{T} \tag{90}$$

Noting that for $t = T$ we return back to $J$ with $\theta = 2\pi$.

Also since the velocity,

$$v = \sqrt{\mu(\frac{2}{r} - \frac{1}{a})} \tag{91}$$

and $r = a$ the velocity is uniform throught out the Circular Orbit. So (90) holds.

For non circular orbits, the velocity is changing with position. In the following we will outline how to predict position, with specified time.

Now define $n$, the mean motion, as,

$$n = \sqrt{\frac{\mu}{a^3}} \tag{92}$$

So base on (89),

$$n = \frac{2\pi}{P} \tag{93}$$

Figures 13 and 14 show the auxillary circle and the eccentric anomaly $E$ for two diffrenet positions $\theta$.

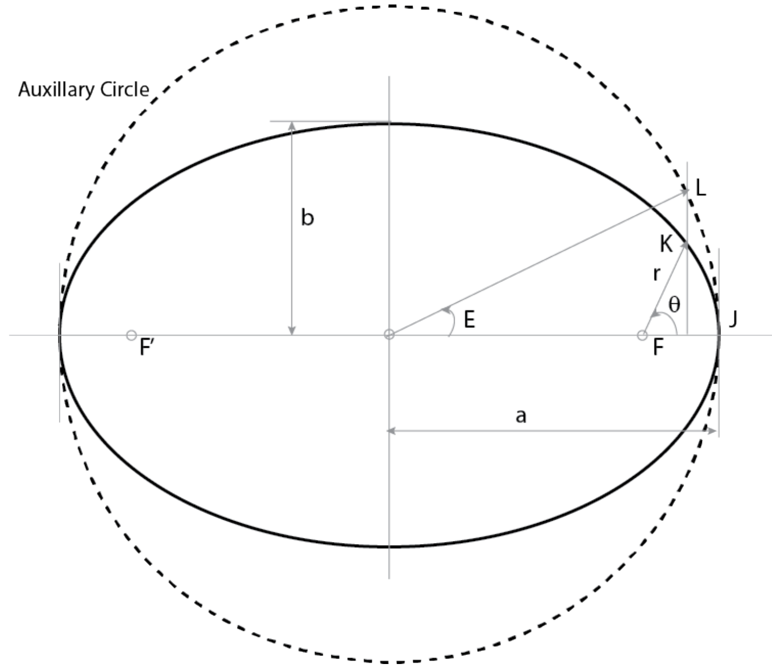


Figure 13: Elliptic Circular Orbit With Auxillary Circle and Showing Eccentric Anamoly, E

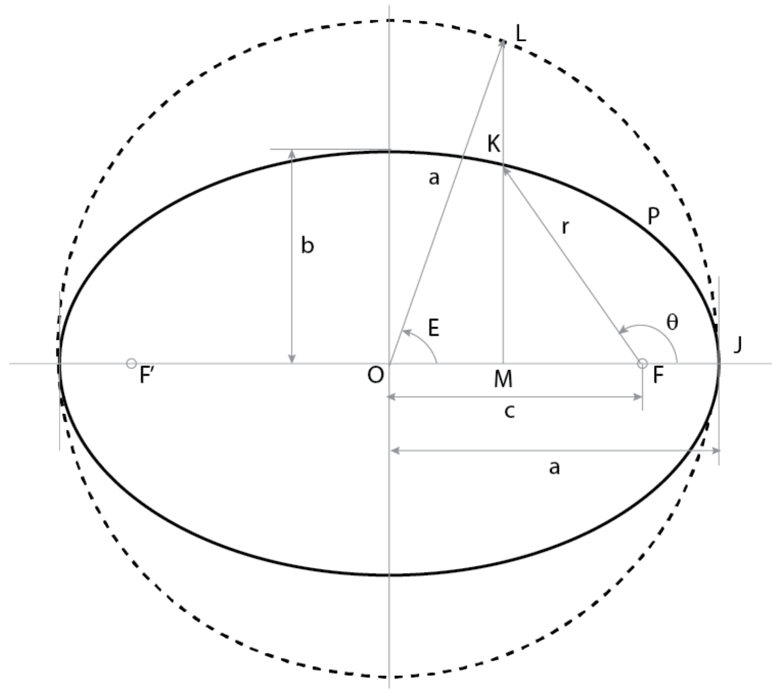


Figure 14: Elliptic Circular Orbit With Auxillary Circle and Showing Eccentric Anamoly, E Another Location

In the developments to come, we will need to express the eccentric anomaly $E$ in terms of $\theta$ and vice versa. First lets relate $E$ to $\theta$.

In Figure 14,

$$\cos(E) = \frac{OM}{OL} = \frac{OF - MF}{OL} = \frac{c + r\cos(\theta)}{a} \tag{94}$$

Or in terms of the eccentricity $e = \frac{c}{a}$, and noting that $r = \frac{p}{1+e\cos(\theta)}$ and that $p = a(1-e^2)$,

$$\cos(E) = \frac{e + \cos(\theta)}{1 + e\cos(\theta)} \tag{95}$$

## 5.1 Analytical Method

See [1] and [7] for the Analytic Method for determining position based on time of flight. We shall derive the analytical equation by following [1]. We use our notation.

Recall from equation (71)

$$dt = \frac{r^2 d\theta}{H} \tag{96}$$

Integrating,

$$\int_T^t H dt = \int_0^\theta r^2 d\theta \tag{97}$$

Or,

$$(t - T)H = \int_0^\theta \frac{p^2 d\theta}{[1 + e\cos(\theta)]^2} \tag{98}$$

Now a change of variables following [1] to the eccentric anomaly $E$ is in order. Recall that from (99),

$$\cos(E) = \frac{e + cos(\theta)}{1 + e\cos(\theta)} \tag{99}$$

the following relations are established between $\theta$ and $E$:

$$\cos(\theta) = \frac{e - cos(E)}{e\cos(E) - 1} \tag{100}$$

and,

$$\sin(\theta) = \frac{a\sqrt{1 - e^2}}{r}\sin(E) \tag{101}$$

Substituting for $\cos(\theta)$ using (100) noting that $r = \frac{p}{1+e\cos(\theta)}$

Now to obtain the relationship between $E$ and $r$ just like we have the relationship between $r$ and $\theta$, we proceed as outlined in [6].

We start from (94) which we show below:

$$\cos(E) = \frac{c + r\cos(\theta)}{a} \tag{102}$$

Then,

$$r = \frac{a\cos(E) - c}{\cos(\theta)} \tag{103}$$

$$r = \frac{a\cos(E) - c}{\frac{\cos(E) - e}{1 - e\cos(E)}} = \frac{(a\cos(E) - c)[1 - e\cos(E)]}{\cos(E) - e} \tag{104}$$

$$r = \frac{a(\cos(E) - e)[1 - e\cos(E)]}{\cos(E) - e} \tag{105}$$

Finally,

$$r = a[1 - e\cos(E)] \tag{106}$$

Differentiating (100 ) we obtain,

$$d\theta = \frac{\sin(E)[1 + e\cos(\theta)]}{\sin(\theta)[1 - e\cos(E)]}dE = \frac{\sin(E)(\frac{p}{r})}{\sin(\theta)\frac{r}{a}}dE \tag{107}$$

$$d\theta = \frac{a\sqrt{1 - e^2}}{r}dE \tag{108}$$

Where we substituted for $\sin(\theta)$ using (101) in (107).

Then,

$$(t - T)H = \frac{p}{\sqrt{1 - e^2}}\int_0^E rdE \tag{109}$$

$$(t - T)H = \frac{pa}{\sqrt{1 - e^2}}\int_0^E [1 - e\cos(E)]dE \tag{110}$$

$$(t - T)H = \frac{pa}{\sqrt{1 - e^2}}[E - e\sin(E)] \tag{111}$$

Since $H = \sqrt{\mu p}$,

$$(t - T) = \frac{a^3}{\mu}[E - e\sin(E)] \tag{112}$$

This equation is referred as Kepler's equation. This equation can be solved for $E$ given $t$ using Newton-Raphson successive Approximation. There is no closed-form solution (never dispair!). In the next section we present Mathematica® code (Open Source, GPL3) for solving for Position as a Function of Time. The key equations are summarized below.

$$(t - T) = \frac{a^3}{\mu}[E - e\sin(E)] \tag{113a}$$

$$\cos(\theta) = \frac{e - cos(E)}{e\cos(E) - 1} \tag{113b}$$

$$\sin(\theta) = \frac{a\sqrt{1 - e^2}}{r}\sin(E) \tag{113c}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \tag{113d}$$

$$\theta = \tan^{-1}(\theta) \tag{113e}$$

### 5.2 Mathematica® Code Orbit Position as a Function of Time for Elliptic Orbits

For the latest updates to the Mathematica® code visit:

```
https://scientificworks.org/orbital_mechanics.html
```

In the code, the following parameters were used [6]:

$$\mu = 3.986004415 * 10^5 \quad (114a)$$
$$e = 0.6 \quad (114b)$$
$$p = 20,410\ km \quad (114c)$$
$$a = 31890\ km \quad (114d)$$
$$b = 25512\ km \quad (114e)$$
$$r_p = 12756\ km \quad (114f)$$
$$r_a = 51024\ km \quad (114g)$$
$$c = 19134\ km \quad (114h)$$
$$EarthRadius = 6371\ km \quad (114i)$$

Constants

$$\mu = GM \quad (115a)$$
$$G = 6.67430 \times 10^{-11} N - m^2 \cdot kg^{-2} \quad (115b)$$
$$M_{Earth} = 5.9722 \times 10^{24} kg \quad (115c)$$
$$\mu_{Earth} = 3.986004415 \times 10^5 \frac{km^3}{s^2} \quad (115d)$$

The results are shown in Figures 15, 16, and 17. Note that in both Figures 16 and 17 the velocity decreases as time increases from (t0) Perigee to Apogee t=T.

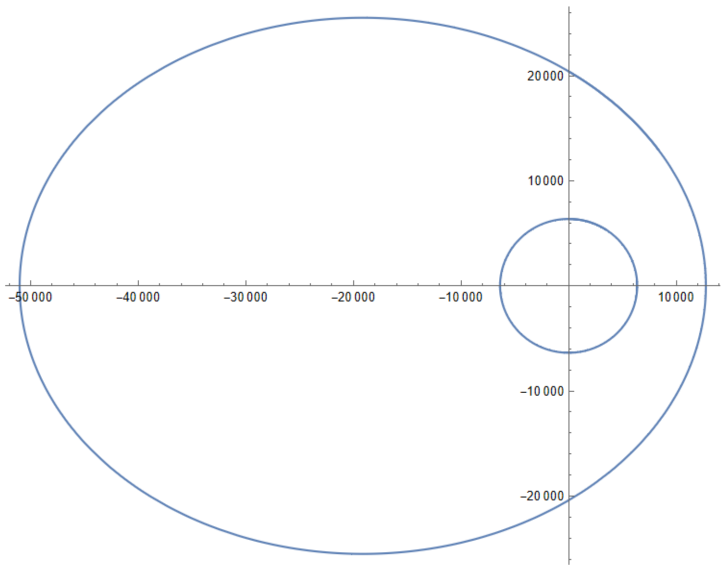


Figure 15: Elliptic Orbit Mathematica® Code

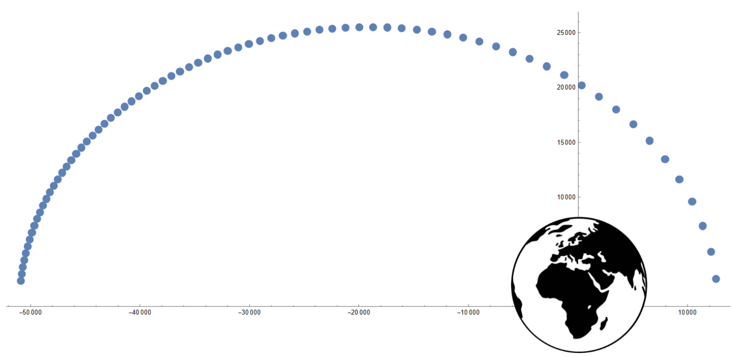

Figure 16: Position $\theta$ True Anomaly Orbit as Time is Varied for 7.5 Hours with Period = 15.74 Hours Mathematica® Code

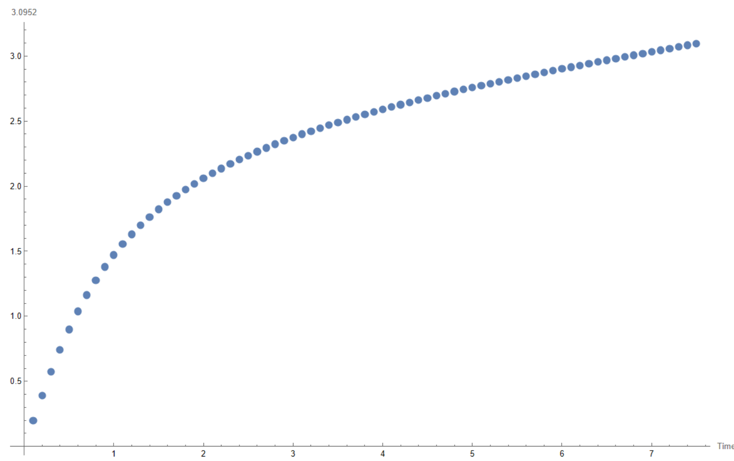


Figure 17: Position $\theta$ True Anomaly as Time is Varied from 0 to 7.5 Hours in 0.1 Hour Steps with Period = 15.74 Hours Mathematica® Code

Mathematica® Code Orbit Position as a Function of Time for Elliptic Orbits

```
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(* Orbit Position as a Function of Time for Elliptic Orbits  *)
(* Orbital Mechanics                                                *)
(* Author:Sasan Ardalan                                          *)
(* ScientificWorks.org                                              *)
(* Date:September 10,2024                                      *)

(* Parameters from:                                            *)
(* Fundamentals of Astrodynamics and Applications,*)
(*  4th Edition, Vallado                                         *)
(* Earth Orbit                                                      *)

mu = 3.986004415*10^5;

e = 0.6;
p = 20410;
a = 31890;
b = 25512;
rp = 12756;
ra = 51024;
c = 19134;
earthRadius = 6371;

Print["Period is:"]
period = 2*Pi*Sqrt[a*a*a/mu]/3600

Show[PolarPlot[p/(1 + e*Cos[theta]), {theta, 1, 2.5*Pi}],
 PolarPlot[earthRadius, {theta, 1, 2.5*Pi}]]
```

```
SinE[theta_, e_] := Sqrt[1 - e*e] Sin[theta]/(1 + e*Cos[theta])
CosE[theta_, e_] := (e + Cos[theta])/(1 + e*Cos[theta])
SinTheta[Ev_, e_] := (Sqrt[1 - e*e]*Sin[Ev])/(1 - e*Cos[Ev])
CosTheta[Ev_, e_] := (Cos[Ev] - e)/(1 - e*Cos[Ev])

P = 2*Pi*Sqrt[a*a*a/mu]
Phours = P/3600
n = 2*Pi/P

satPosTime[th_, p_, e_, n_] :=
 Module[{t, t0, M0, EvRoot, T, M, Ev, thetaval}, t = th*3600; t0 = 0;
  T = 0; M0 = n*(t0 - T); M = n*(t - t0) + M0;
  EvRoot = Ev /. FindRoot[M == Ev - e*Sin[Ev], {Ev, M0}];
  sinTheta = SinTheta[EvRoot, e];
  cosTheta = CosTheta[EvRoot, e];

  thetaval = ArcTan[sinTheta/cosTheta];
  thetaval = If[thetaval < 0, thetaval + Pi, thetaval];
    Return[thetaval]  ]

Print["Sat Positione Time:"]
satPosTime[0.1, p, e, n]*180/Pi

satpos[th_, p_, e_, n_] := p/(1 + e*Cos[satPosTime[th, p, e, n]]);

Print["Sat Radius Time:"]
th1 = 7
satPosTime[th1, p, e, n]*180/Pi
satpos[th1, p, e, n]
```

```
track = {, }
path = {, }
th1 = 0
delta = 0.1;
For[i = 0, i < 75, i++, th1 = th1 + delta;
         theta = satPosTime[th1, p, e, n];
          r = satpos[th1, p, e, n];
          velocity = Sqrt[mu*(2/r - 1/a)];
          (* track=Append[track,{theta,velocity}]; *)
          (*track=Append[track,{th1,velocity}];*)

          track = Append[track, {th1,  theta  }];

          path = Append[path, {theta, r}];

         (*Print[r,":",theta*180/Pi,":",velocity];*)
 ]
(*Print ["The Track"]
Print[track]*)
ListPlot[track, AxesLabel -> {Time, theta}]
(* Print ["The Path"]
Print[path] *)
ListPolarPlot[path]
```

# Chapter 6

## Plotting the Velocity Vector in Mathematica®

It is important to visualize the velocity vector as function of tue anomaly ($\theta$). For this purpose Mathematica® code was written. For reference, Figure 18 shows the orbit with Earth. This is the same as in Figure 15. The results for the parameters in [6] are shown in Figures 19.

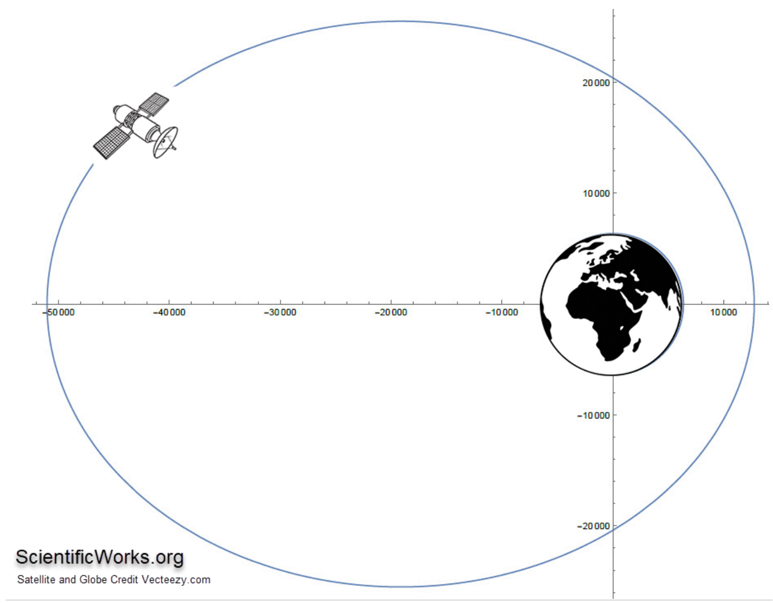


Figure 18: Elliptic Orbit Mathematica® Code

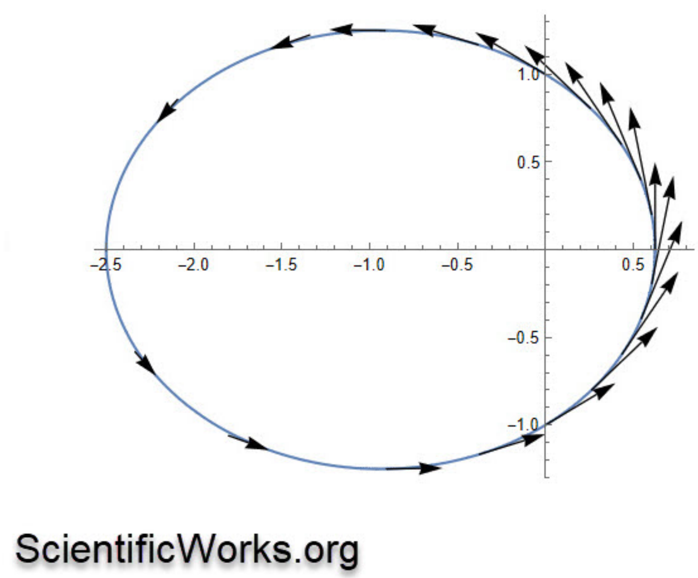


Figure 19: Velocity Vector Mathematica® Code

The Mathematica® code is provided in the following link:

```
https://scientificworks.org/orbital_mechanics_velocity_vector.html
```

The code is Open Source and Licensed under GPL3.

## Bibliography


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[2] Vladimir A. Chobotov. In *Orbital Mechanics, Third Edition*. The American Institute of Aeronautics and Astronautics, Inc., 2002.

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# Chapter 7

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